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UNIFORM CYCLIC EDGE CONNECTIVITY IN CUBIC GRAPHS

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A cubic graph which is cyclically k-edge connected and has the further property that every edge belongs to some cyclic k-edge cut is called uniformly cyclically k-edge connected ($U(k)$). We classify the $U(5)$ graphs and show that all cyclically 5-edge connected cubic graphs can be generated from a small finite set of $U(5)$ graphs by a sequence of defined operations.

1. Introduction

All graphs considered in this paper shall be undirected simple graphs without loops or multiple edges. Furthermore, unless otherwise stated, we shall assume that they are cubic, that is every vertex has degree three. We shall denote by $V(G)$ the vertex set of the graph G and by $E(G)$ the edge set of G. The subgraphs considered in this paper are induced subgraphs and so, if A and B are (induced) subgraphs, we will use $A \cup B$ to denote the subgraph of G induced by $V(A) \cup V(B)$. The set of edges with precisely one end vertex in a cycle C will be called *cocycle* of C.

A cubic graph G is said to be *cyclically k-edge connected*, $k \geq 3$, if G is 3 edge connected and there is no set E of $k-1$ or fewer edges such that $G - E$ has at least two connected components containing cycles. If G is such a graph and $e = (x, y)$ is an edge in G, form G_e from G by deleting e and suppressing the resulting degree two vertices x, y. This operation is called *removal of an edge*, and is e is *removable* if G_e is also cyclically k-edge connected.

In [1], Andersen, Fleischner and Jackson showed that a cyclically 4-edge connected graph with twelve or more vertices has at least $(|E(G)| + 12)/5$ removable edges. Wormald [4] has shown that every cyclically 4-edge connected cubic graph on ten vertices contains a removable edge. So apart from the graphs on eight vertices, there are no cyclically 4-edge connected cubic graphs without removable edges.

When we consider cyclically 5-edge connected cubic graphs, the situation becomes quite different. Here it becomes apparent that there are infinitely many graphs which have no removable edges. For example, there is the dodecahedron, which we shall denote by DD (Figure 1). If we delete a 5-cycle from this graph, what remains consists of a 5-cycle surrounded by five other 5-cycles in planar fashion. We call this particular configuration a *rosette* (Figure 1). Replacing every vertex in a 5-connected

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5-regular graph by a rosette, results in a cubic graph in which every edge lies in the cocycle of a 5-cycle and therefore lies in a cyclic 5-edge cut. There cannot be a cyclic k-edge cut with $k \leq 4$ since our original graph was 5-connected. Thus this graph has no removable edges. There are clearly infinitely many such graphs.

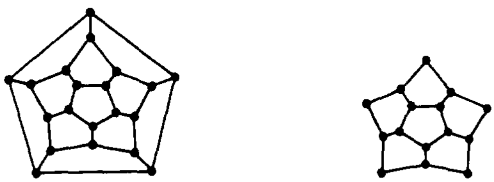


Fig. 1

We say that a cyclically k-edge connected cubic graph G is *uniformly cyclically k-edge connected* (denoted $U(k)$) if and only if there are no removable edges in G. Note that an edge in G is removable if and only if it does not lie in a cyclic k-edge cut, and so G is $U(k)$ if and only if every edge lies in a cyclic k-edge cut.

In this paper we shall classify the $U(5)$ graphs, We shall also show that all cyclically 5-edge connected cubic graphs can be generated from one of a small set of such graphs by a finite sequence of operations of a certain type. This complements the result of Barnette [2] who showed that all planar cyclically 5-edge connected cubic graphs can be generated from the dodecahedron by a finite sequence of similar operations.

2. More $U(5)$ graphs

The rosette we have encountered above is an extremely important and versatile "building block" in the construction of $U(5)$ graphs. We can take an arbitrary tree, with all interior vertices of degree three, and replace the endvertices with rosettes. Joining the rosettes in a single circuit with two edges between each adjacent pair in the circuit gives a $U(5)$ graph.

An arbitrary number of rosettes can be strung together in a ring, each joined to its two neighbours by two edges apiece leaving one degree two vertex in each rosette. With this procedure, we can generalize the example of a 5-regular 5-connected graph and use any k-connected graph with $k \geq 5$, replacing each vertex by a rosette ring of the appropriate size.

These are by no means the only $U(5)$ graphs known. Using the rosette in different ways readily produces new families of $U(5)$ graphs. The question we have asked is whether the rosette is the only means of producing such graphs? The answer to the question is no. There is of course the Petersen graph (P) which has too few vertices to possibly contain a rosette. Checking the lists of twelve vertex cubic graphs in [3] turns up two new $U(5)$ graphs without rosettes (see Figure 2).

Consider the family of graphs defined below.

Definition. An *odd double ladder* of length k is a graph G with vertex set $V(G) = \{a_i : 1 \leq i \leq k\} \cup \{b_i : 1 \leq i \leq 2k+1\} \cup \{c_i : 1 \leq i \leq k+1\}$ and edge set $E(G) = \{(a_i b_{2i}) : 1 \leq i \leq k\} \cup \{(c_i b_{2i-1}) : 1 \leq i \leq k+1\} \cup \{(a_i a_{i+1}) : 1 \leq i \leq$

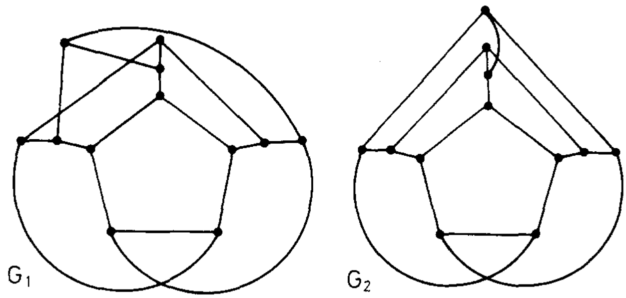


Fig. 2

$k-1\} \cup \{(b_i b_{i+1}) : 1 \le i \le 2k\} \cup \{(c_i c_{i+1}) : 1 \le i \le k\} \cup$ (an arbitrary matching between $\{a_1, b_1, c_1\}$ and $\{a_k, b_{2k+1}, c_{k+1}\}$).

An *even double ladder* of length k is a graph G with vertex set $V(G) = \{a_i : 1 \le i \le k\} \cup \{b_i : 1 \le i \le 2k\} \cup \{c_i : 1 \le i \le k\}$ and edge set $E(G) = \{(a_i b_{2i}) : 1 \le i \le k\} \cup \{(c_i b_{2i-1}) : 1 \le i \le k\} \cup \{(a_i a_{i+1}) : 1 \le i \le k-1\} \cup \{(b_i b_{i+1}) : 1 \le i \le 2k-1\} \cup \{(c_i c_{i+1}) : 1 \le i \le k-1\} \cup$(an arbitrary matching between $\{a_1, b_1, c_1\}$ and $\{a_k, b_{2k}, c_k\}$).

We make the following observations in Lemma form but omit their proof which can easily be supplied by the reader.

Lemma 1. *An odd double ladder of length k is $U(5)$ if and only if $k \ge 2$ and the matching is $\{(a_1 c_{k+1}), (b_1 b_{2k+1}), (c_1 a_k)\}$, or $k = 3$ and the matching is $\{(a_1 b_7), (b_1 c_4), (c_1 a_3)\}$ or $\{(a_3 b_1), (b_7 c_1), (c_4 a_1)\}$, or $k = 4$ and the matching is $\{(a_1 b_7), (b_1 a_4), (c_1 c_5)\}$.*

We refer to the first class of $U(5)$ odd double ladders as *standard* odd double ladders. The standard odd double ladder with $k = 2$ is P. The special case for $k = 3$ is called the *twisted double ladder* denoted TDL. The $k = 4$ special case is called the *twisted dodecahedron*, denoted TDD, and has every 5-cycle in a rosette.

Lemma 2. *An even double ladder is $U(5)$ if and only if $k \ge 5$ and the matching is $\{(a_1 a_k), (b_1 b_{2k}), (c_1 c_k)\}$.*

We call the $U(5)$ even double ladders *standard* even double ladders. The $U(5)$ even double ladder with $k = 5$ is the dodecahedron.

We note that all $U(5)$ double ladders have every edge in the cocycle of a 5-cycle. Not all $U(5)$ graphs have this property.

3. More about $U(5)$ graphs

One of the main results in this paper shows that apart from the two twelve vertex $U(5)$ graphs, the $U(5)$ double ladders are the only $U(5)$ graphs which do not contain rosettes. This result is obtained by investigating the way in which 5-cycles are distributed in a $U(5)$ graph. To show that this is pertinent to all $U(5)$ graphs we first show that all $U(5)$ graphs do indeed contain 5-cycles. We do this by establishing the following stronger result.

Theorem 1. *Let G be a cubic, 3-connected, cyclically k-edge connected graph with the additional property that there exists a cyclic k cut K such that $G - K$ has one component of minimum size and there is an edge e in this minimum component which lies in a cyclic k cut K'. Then G has girth k.*

Proof. Let G be a graph of the type described above. Then G has the form shown in Figure 3 where the component $A \cup B$ of $G - K$ is minimum. The edges emanating from any of the subgraphs A, B, C, D belong to K, K' or both.

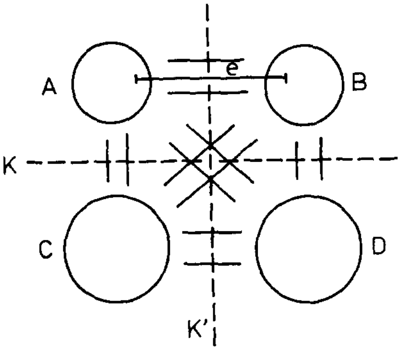


Fig. 3

Now $|K| + |K'| = 2k$. Hence, if A (B) is incident with $k+1$ or more edges of $K \cup K'$, then D (C) is incident with at most $k-1$ edges of $K \cup K'$. Since G is cyclically k-edge connected, D (C) must be acyclic and contain fewer vertices than A (B). This contradicts the minimality of $A \cup B$. Thus we may assume that each of A and B has k or fewer edges emanating from it and hence that they are both acyclic.

We shall denote by $e(X)$ the number of edges emanating from the subgraph X and by $c(X)$ the number of connected components in X.

Using this terminology the above observation is that

$$e(A) \le k \qquad \text{and} \qquad e(B) \le k. \tag{1}$$

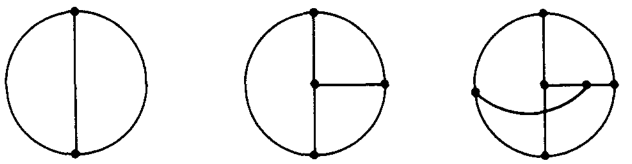
Considering the acyclic nature of A and B we obtain the following expressions for $e(A)$ and $e(B)$.

$$e(A) = |V(A)| + 2c(A), \qquad e(B) = |V(B)| + 2c(B). \tag{2}$$

Combining the information in (1) and (2) above we obtain

$$|V(A)|+|V(B)|=e(A)+e(B)-2(c(A)+c(B)) \le 2k-2(c(A)+c(B)) \le 2k-4. \tag{3}$$

The subgraph $A \cup B$ is 2-connected and contains exactly k vertices of degree two. All other vertices in $A \cup B$ have degree three. Let C be a cycle of minimum length in $A \cup B$. Note that C must have length at least k because G is cyclically k-edge-connected. If C has length greater than k, then it includes vertices with degree three in $A \cup B$. There the evenly many such vertices in $A \cup B$, at least two of which must lie on C if it has length greater than k. If there are two degree three vertices, then



(i) (ii) (iii)

Fig. 4

$k \geq 6$ (by (3)) and we have the situation indicated in Figure 4(i) (where degree two vertices are suppressed). In this case there are three cycles; each degree two vertex appearing in two cycles and the degree three vertices appearing in all three. Thus the average length of these cycles is $(2k+6)/3 \leq k$.

If there are four degree three vertices, then $k \geq 8$ and we have the situation shown in Figure 4(ii) (since two disjoint cycles must give one of length less than k). In this case there are seven cycles with an average length of $(4k+24)/7 \leq k$.

If there are six degree three vertices, then $k \geq 10$ and we have the situation shown in Figure 4(iii). In this case there are fourteen cycles with an average length of $(7k+66)/14 \leq k$.

There cannot be eight or more degree three vertices without having two disjoint cycles at least one of which must be of length less than k.

The above shows that G has girth $\leq k$ and by definition such a graph must have girth $\geq k$. Thus we have established the result. ∎

Corollary. *A $U(k)$ graph has girth k.* ∎

4. Classification of $U(5)$ graphs

We are now in a position to classify the $U(5)$ graphs with the following theorem.

Theorem 2. *Let G be a $U(5)$ graph which contains a 5-cycle which does not belong to a rosette. Then G is either a double ladder or one of G_1, G_2 (see Figure 2).*

Proof. Let $C = (1,2,3,4,5,1)$ be a 5-cycle in G that does not belong to a rosette. As G is $U(5)$, each edge in C belongs to a cyclic 5-edge cut (as these are the only types of cuts to which we shall make reference, we shall usually refer to them as "5-cuts" or simply "cuts"). A 5-cut which has an edge in common with C has exactly two independent edges in common with C. From this observation we see that there are 5-cuts K and K' which intersect C as shown in Figure 5. Note that the edge f may be contained in K', and g may be contained in K, but $(3,4)$ cannot be in either K or K'.

We consider three cases separately.

Case 1: $f \in K'$ and $g \in K$.

Case 2: $f \in K'$ and $g \notin K$ ($f \notin K'$ and $g \in K$ follows by symmetry from this case).

Case 3: $f \notin K'$ and $g \notin K$.

Case 1: $f \in K'$ and $g \in K$. So G has the form shown in Figure 6. If there is an edge from A to C, then there are also edges from A to B and from B to C. Hence $|V(B)| = 1$, $|V(A)| = 2$ and $|V(C)| = 2$ (by the cyclic connectivity constraints).

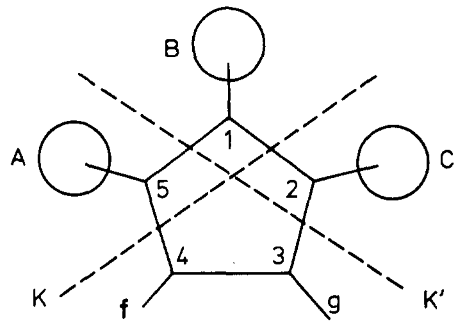


Fig. 5

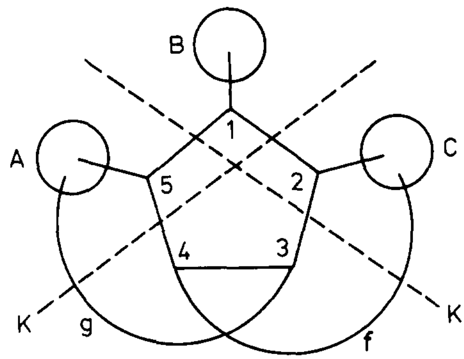


Fig. 6

Consequently, G is the Petersen graph, the smallest standard odd double ladder. Thus we assume that there is no edge from A to C.

The cuts K' and K are completed by two edges from A to B and two edges from B to C respectively. Consequently both A and C are isomorphic to K_2. The edge $(3,4)$ lies in the intersection of three 5-cycles which means that a 5-cut containing $(3,4)$ must also contain one further edge from each of these three 5-cycles. The fifth edge from such a cut must disconnect the subgraph B. Cyclic connectivity requires that the two resulting components of B are an edge and an isolated vertex respectively. Hence B consists of a path on three vertices. The total number of vertices in the resulting graph is twelve and G is isomorphic to one of G_1, G_2. This completes Case 1.

Case 2: $f \in K'$ and $g \notin K$. In this case G is as shown in Figure 7. Edges joining pairs of subgraphs from A, B, C, D must contribute to K, K' or both. If a subgraph is acyclic, then not all of the edges joining this subgraph to the remainder of G belong to K, and not all belong to K'. In particular, we notice that if A, B, C or D is acyclic, then each endvertex of that subgraph is incident with one edge from each of K and K' in G.

As we noted in Case 1, an edge from A to C contributes to both K and K', so that we have three edges specified in K and four in K'. In order that 3-connectivity be maintained, each of B and D must have at least two further edges joining them

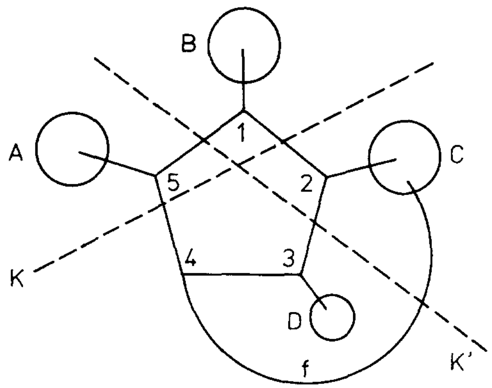


Fig. 7

to the rest of G. These contribute at least four more edges to $K \cup K'$ which can have at most three more edges. This implies that there is no edge from A to C.

We argue similarly in the case where there is an edge from B to D and find that preserving 3-connectivity, $|K|$ and $|K'|$, we must have single edges from A to D, from A to B and from B to C. This forces A, C and D to be single vertices and B to be a single edge, thereby inducing a 4-cycle $(C, 2, 3, 4, C)$ in G. Thus we conclude that there is no edge from B to D.

If there are two edges between A and B, then there cannot be an edge from D to C. Thus all of the edges from D must contribute to K. This means that there are only four edges joining D to the remainder of G and thus D is acyclic. This is impossible, as was observed above. So K' is completed by one edge from A to D and one edge from C to D. K is completed either by two edges from A to B and one edge from B to C, or by one edge from A to D and two edges from B to C.

In the former case A, C and D are single edges and B is a single vertex. This forces a 4-cycle in $A \cup D$.

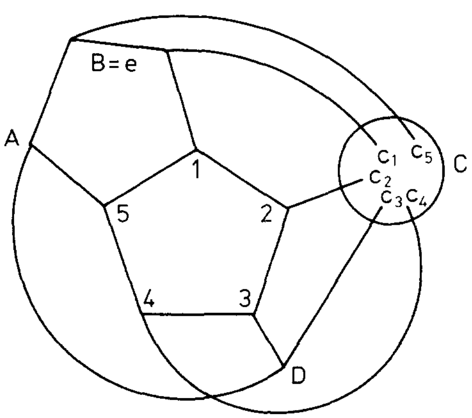


Fig. 8

In the latter case, A and D are single vertices, B is a single edge and G has the form shown in Figure 8.

In this graph, the indicated edge e must belong to a 5-cut κ which also contains one of $(A, 5)$ and $(1, 5)$. The 5-cycle $(A, 5, 4, 3, D, A)$ cannot lie in a rosette and thus, by symmetry, we may assume that κ contains $(1, 5)$. This cut must also contain either $(3, 4)$ or $(2, 3)$.

If we have $(3, 4)$, then we must also have (A, D) and one further edge which disconnects C. This means that C consists of a path on three vertices and that G is isomorphic to one of G_1, G_2.

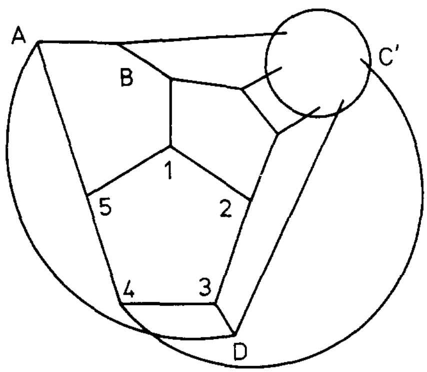


Fig. 9

Thus we assume that κ contains e, $(1, 5)$, $(2, 3)$ and two further edges which disconnect C, and thus separate c_1, c_2 from c_3, c_4, c_5. So κ is the cocycle of a 5-cycle and G has the form shown in Figure 9. In this graph we consider a 5-cut that contains the edge $(3, 4)$. Such a 5-cut must also contain a further edge from each of the 5-cycles $(3, 4, 5, A, D, 3)$ and $(1, 2, 3, 4, 5, 1)$. As both of the edges $(1, 2)$ and $(1, 5)$ lie in other 5-cycles, whichever of these edges we include in our 5-cut, we shall require at least one more edge from those drawn explicitly in Figure 9. Consequently this 5-cut must be completed by one more edge which disconnects the subgraph C'. The resulting graph has fourteen vertices and is an odd double ladder.

It remains only for us to consider the third of our three cases.

Case 3: $f \notin K'$ and $g \notin K$, which gives G the structure shown in Figure 10. As G has girth five, the subgraph D contains more than a single edge and so must be connected to the rest of G by at least five edges. We shall first consider the possibility that there is an edge from D to B.

An edge from D to B contributes to both K and K' and so there can be at most one such edge or 3-connectivity will be violated at A or C. Conserving 3-connectivity and observing that the edges in a cyclic edge cut are independent, forces structure of G to be equivalent to that in Figure 8 which we have already considered.

In view of the above, we assume that there is no edge from B to D in the graph of Figure 10. Consequently, there are at least three edges in G from D to either A or C. Without loss of generality, we may assume that there are at least two edges from D to A and at least one edge from D to C. A third edge from D to A would complete K and force all further edges from B and C to contribute only to K'. This

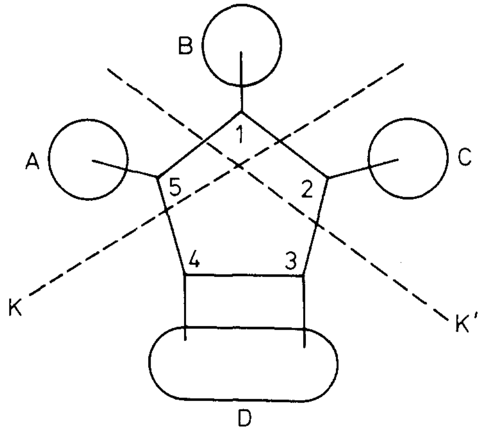


Fig. 10

would violate the 3-connectivity of G. Thus we conclude that there are exactly two edges from D to A. In a similar way we can rule out the possibility of an edge from A to C. The only possible way to complete K is with a single edge from B to C. The two possible ways in which the cut K' can be completed are: (i) with a single edge from A to B and a second edge from D to C, or (ii) with two edges from A to B. Both of these possibilities are shown in Figure 11.

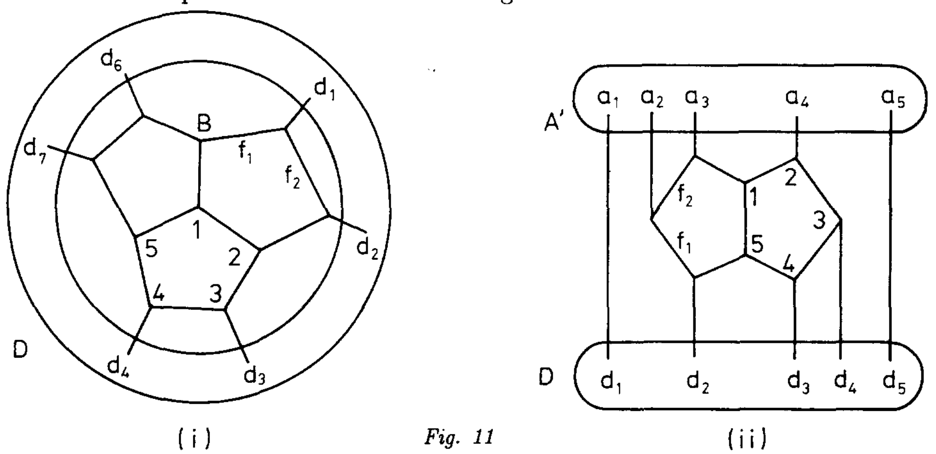


Fig. 11

Let us consider first the graph in Figure 11(i). A 5-cut including the edge $(3,4)$ must also include one of the edges in $\{(1,5),(1,2)\}$. If the 5-cut contains $\{(3,4),(1,2)\}$, then we have three edges with which to disconnect both subgraphs A and D. Thus one of the subgraphs is disconnected by the removal of a single edge. In A we must separate $\{a_1,a_2,a_3\}$ from $\{a_4,a_5\}$ and in D we must separate $\{d_1,d_2,d_3\}$ from $\{d_4,d_5\}$. Separating A in this way with a single edge would result in a 4-cycle so we conclude that D is the subgraph which contains only one more edge from the cutset in question. There is just one way in which D can be specified so as to satisfy the above and that is shown in Figure 12. But clearly this is a graph of the type in Figure 9 considered earlier.

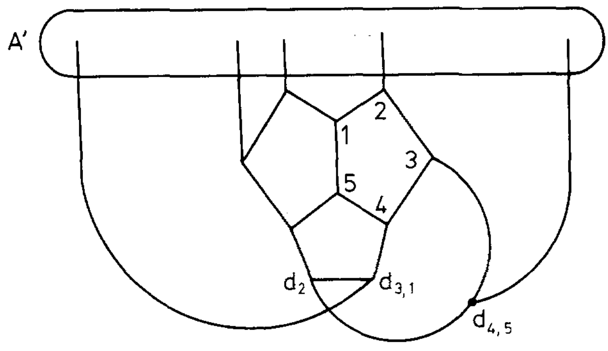


Fig. 12

Finally we assume that the 5-cut contains $\{(3,4),(5,1)\}$. Then we must also have one of $\{f_1, f_2\}$. If we have $\{(3,4),(5,1), f_2\}$ in our 5-cut, then the graph is completely specified and is the graph in Figure 13. In this graph, however, there are two removable edges (shown as g_1 and g_2). Thus G is not $U(5)$. The last possibility in this case is a 5-cut which contains $\{(3,4),(5,1), f_1\}$. This results in the cocycle of a 5-cycle which places us in the same situation as Figure 11(ii) which we shall now consider.

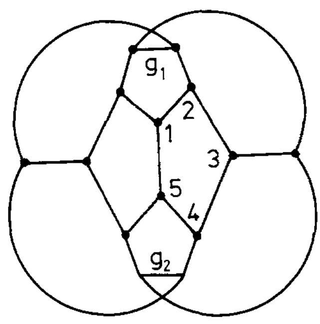


Fig. 13

Since we assume that every edge lies in a 5-cut, the edge $(3,4)$ must lie in such a cut. Without loss of generality, the cut must also contain the edge $(1,2)$. In addition to $\{(1,2),(3,4)\}$ we must also include one of $\{f_1, f_2\}$.

Including the edge f_1 requires that there are two edges in D which separate $\{d_1, d_2, d_3\}$ from $\{d_4, d_5, d_6\}$. This gives the graph in Figure 14. A 5-cut in this graph including the edge f_2 must also include $h \in \{(1,2),(1,B)\}$ and a second edge k from the other 5-cycle containing h. Unless k is $(3,4)$, the cut must be completed by one further edge from each of D' and D''. Thus each of D' and D'' must be a path on three vertices and G is completely specified. It is impossible, however, for this graph to be $U(5)$ (the only graphs obtainable by replacing D' and D'' by three vertex paths either have girth less than five or contain removable edges). We conclude from this that edge $(3,4)$ is included in the cut and that it is the cocycle of a 5-cycle. By symmetry we see that the only 5-cut including the edge f_3 is also the cocycle of a 5-cycle. Each of the new 5-cycles we have found in the above have edges

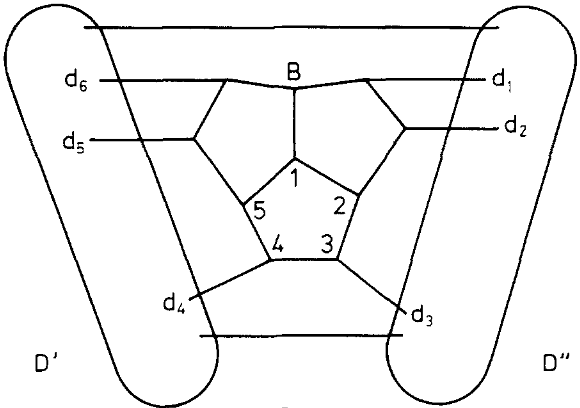


Fig. 14

which are not in any explicitly drawn 5-cut. Using similar arguments to the above we see that these edges can only be included in 5-cuts which are the cocycles of 5-cycles. Continuing in this way we extend a double ladder like structure of 5-cycles as in Figure 15. When we have grown this ladder as far as it goes, there is no way in which the graph can be completed in $U(5)$ fashion.

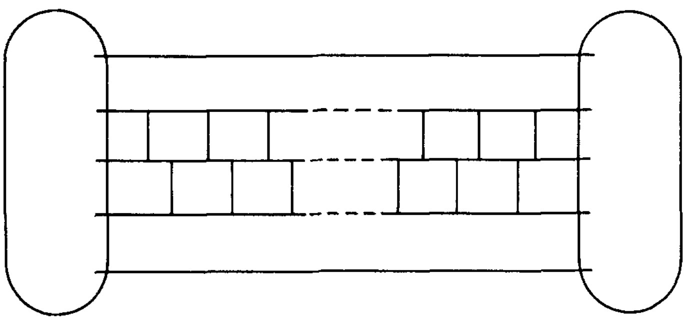


Fig. 15

If we include the edge f_2, then we must disconnect D with two edges which separate $\{d_1, d_4, d_5, d_6\}$ from $\{d_2, d_3\}$. This forces the cut to be the cocycle of a 5-cycle and G must be of the form shown in Figure 16. Following a similar course of argument to the above we find that the only possible 5-cuts containing the indicated edge in Figure 16 are cocycles of 5-cycles. The resulting graph is shown if Figure 17.

Again we proceed as above to find a 5-cut which includes the indicated edge e. Here we find that there are two possible 5-cuts; both of them cocycles of 5-cycles. However, only one of these 5-cycles can occur as we chose the original 5-cycle, C, not to lie in a rosette. Thus G is as shown in Figure 18. This graph consists of an unspecified "blob", D, and a "double ladder" of 5-cycles. There are two edges in the "double ladder" which do not lie in specifically drawn 5-cycle cocycles. If we are to include these edges in 5-cuts, then we have no choice other than to have them in the cocycles of 5-cycles too. We can achieve this in one of two ways:

(i) Extend the already existing "double ladder" of 5-cycles;

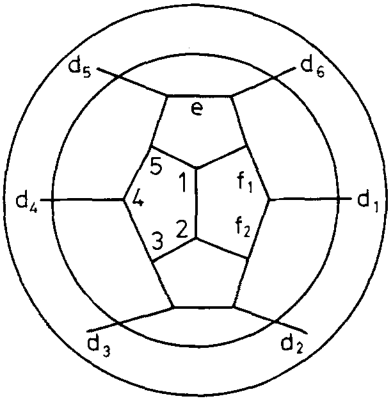


Fig. 16

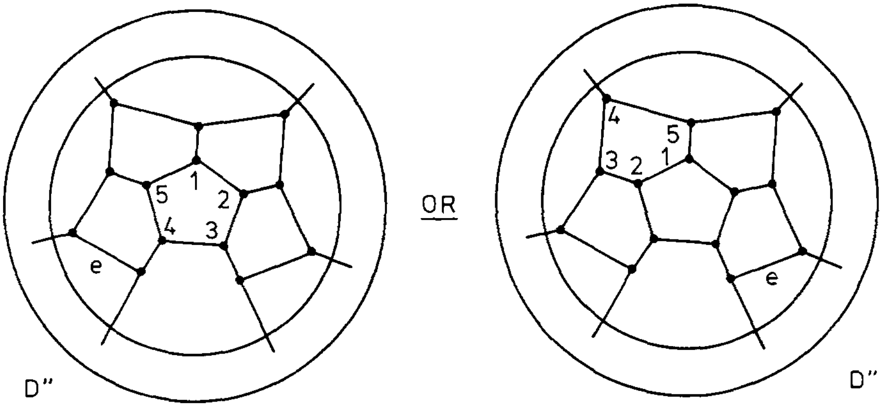


Fig. 17

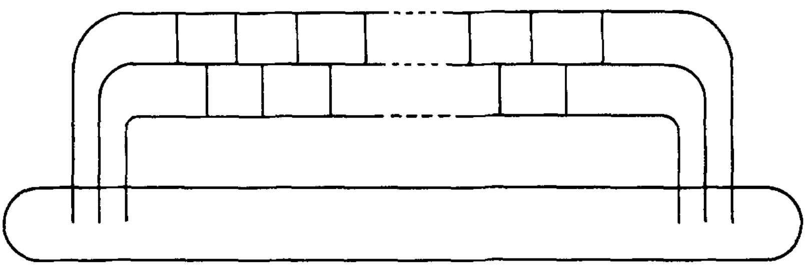


Fig. 18

(ii) Identify the ends of the "double ladder" in such a way as to maintain the $U(5)$ property.

The first of these options having been performed, we find that there are still two edges that do not lie in specifically drawn cyclic 5-edge cuts. The only choices we have for their cyclic 5-edge cuts are the two given above.

The second option completes the graph in accordance with the statement of the theorem. This completes the proof of the theorem. ∎

5. Generating cyclically 5-edge connected cubic graphs

Barnette [2] showed that all cyclically 5-edge connected cubic planar graphs can be obtained from the dodecahedron by a sequence of operations from a prescribed set. In this section of the paper we describe these operations and prove that all cyclically 5-edge connected cubic graphs can be obtained from a small finite set of graphs by a sequence of operations from those we shall now define.

The first operation is adding an edge. It involves subdividing two edges, introducing a degree two vertex on each of the subdivided edges and adding an edge between these vertices. We shall refer to this as an operation type 1 (Figure 19).

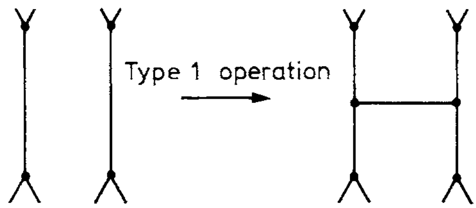


Fig. 19

The second operation is subdividing adjacent 5-cycles. This operation takes two adjacent 5-cycles $(1,2,3,4,5,1)$ and $(1,2,6,7,8,1)$. We subdivide $(1,2)$ and add a degree two vertex u. Then we subdivide $(u,2)$ and add a second degree two vertex v. The edges $(4,5)$ and $(6,7)$ are also subdivided and are given degree two vertices w and x, respectively. The operation is completed by the addition of the edges (u,x) and (v,w). We shall refer to this as an operation of type 2 (Figure 20). (We note that a type 2 operation can be replaced by two type 1 operations but retain the distinction for clarity of exposition.)

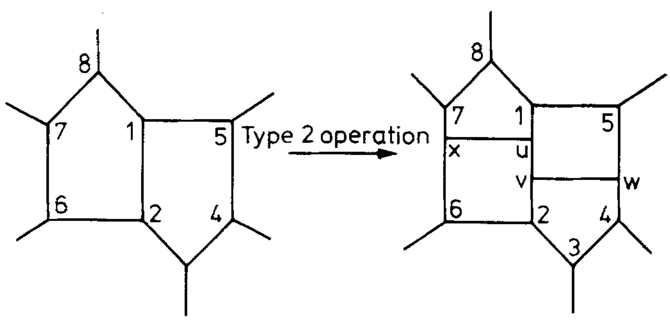


Fig. 20

The third and final operation is inserting a 5-cycle. This is achieved by taking an existing 5-cycle $(0,2,4,6,8,0)$ in G, subdividing each edge $(ii+2(\text{mod } 10))$ and

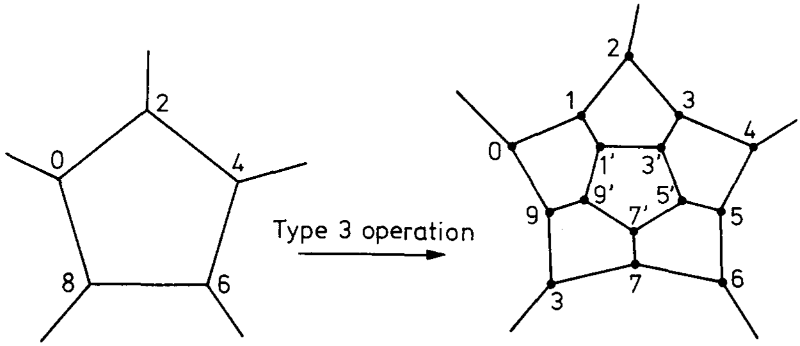


Fig. 21

adding the degree two vertex $i+1$. Each vertex $j = 1, 3, 5, 7, 9$ is then joined to the vertex j' in a new 5-cycle $(1', 3', 5', 7', 9', 1')$. We shall call this an operation of type 3 (Figure 21).

It should be noted that the class of cyclically 5-edge connected graphs is closed under performing operations of types 2 and 3. Performing an operation of type 1 can yield a 3-cycle or a 4-cycle but avoiding these gives closure here too.

Theorem 3. *If G is a cyclically 5-edge connected cubic graph, then G can be obtained from one of the graphs in $S\{P, G_1, G_2, TDL, TDD, DD\}$ by a sequence of operations of types 1, 2 and 3.*

Proof. Let us assume that G is a minimum counterexample to the theorem. Then G must be $U(5)$, otherwise G has a removable edge. The graph G' obtained from G by removing that edge is also cyclically 5-edge connected and by the minimality of G, G' can be obtained from some graph in S by a sequence of allowed operations. But G is obtained from G' by adding an edge (type 1). This contradicts the assumption that G is a counterexample to the theorem.

By Theorem 2, we know that if G contains a 5-cycle not in a rosette, then G is one of the graphs G_1, G_2, P, TDL or a standard double ladder of length $k \geq 3$. G_1, G_2, P, TDL are obtained by a null sequence of operations from graphs in S, so G must be a standard double ladder of length $k \geq 3$. But all standard double ladder are obtained from P or DD by a sequence of type 2 operations.

Finally we must consider the case when G contains a rosette. Let G' be the graph obtained from G by deleting the 5-cycle $(1', 3', 5', 7', 9', 1')$ from the centre of the rosette and suppressing the resulting degree two vertices 1, 3, 5, 7, 9. The outside 10-cycle of the rosette becomes the 5-cycle $(0, 2, 4, 6, 8, 0)$ in G'.

G' is clearly cubic and we claim that it is also cyclically 5-edge connected. To show this we assume that G' contains a cyclic 4-edge cut K (it is clear that G' is at least cyclically 4-edge connected). K must, without loss of generality, contain the edges $(0, 2)$, $(4, 6)$ and two others. Since G is known to be $U(5)$, K must be the cocycle of a 4-cycle and G must have the form shown in Figure 22. A 5-cut in G containing the edge e must, again without loss of generality, also contain $(2, 3)$, $(1, 1')$, $(0, 9)$ and one further edge separating $\{u, v\}$ from $\{w, x, y\}$. Hence G is TDD. But G was chosen to be a counterexample to the theorem so this cannot be. Consequently G' is cyclically 5-edge connected. By the minimality of G, G' can be obtained from some graph in S by a sequence of allowed operations. But G is obtained from G' by an operation of type 3 contradicting the choice of G as a counterexample to the theorem. This completes the proof. ∎

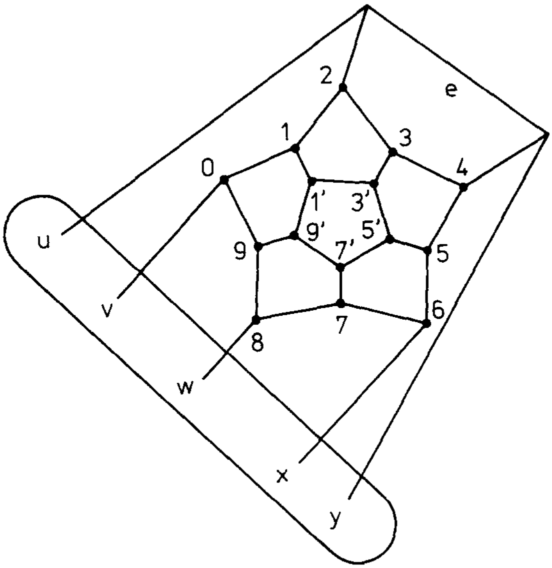


Fig. 22

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